

Spectrum of end states of gravitational collapse with tangential stresses

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The final state—a black hole or naked singularity—of the gravitational collapse of a marginally bound matter configuration in the presence of tangential stresses is classified, in full generality, in terms of the initial data and equation of state. If the tangential pressure is sufficiently strong, configurations that would otherwise evolve to a spacelike singularity result in a locally naked singularity, both in the (initially) homogeneous and in the general, inhomogeneous density case.

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I. INTRODUCTION

The final state of gravitational collapse remains one of the outstanding problems in classical general relativity. For a given set of Cauchy data, hyperbolic evolution of the field equations leads to a unique final state and, under certain conditions, a singularity is formed [1]. However, it is not known whether this singularity is spacelike, or a visible spacetime singularity. Understanding how such singularities arise from regular initial data, and whether they can be visible (at least to local observers), is still to a large extent an open issue [2].

The most studied analytical solution is that describing spherical dust collapse, whose detailed analyses over the past ten years have provided many valuable insights into the formation, visibility, and causal structure of singularities [3]. Albeit of considerable interest in its own right, dust collapse constitutes a highly idealized model, both in terms of geometry and matter content. While the former may arguably be a reasonable approximation for astrophysical collapse [4], the existence of pressures in high density regimes, together with an effective equation of state, cannot be neglected in realistic situations [5]. It would, therefore, be highly desirable to understand the final state of spherical gravitational collapse for generic equations of state. Unfortunately, this objective looks still quite far from being reachable, in particular in the case of perfect fluid sources. Therefore, research has turned to less ambitious objectives. An obvious way to generalize dust (i.e., stress-free) models, is to try to “add” nonvanishing stresses.

A model in which analytical treatment appears feasible is that of vanishing radial stresses, since in this case the general exact solution is known in closed form [6,7] (the opposite case, in which only a radial stress is present, has also been recently considered [8]). The formation and nature of singularities in gravitational collapse with tangential stresses has recently been studied quite extensively [9–12]. In the models analyzed so far, a tendency to *uncover* part of the singularity spectrum was observed, i.e., initial configurations that would

otherwise end up in black holes develop locally naked singularities (in one special case, the singularity was even shown to be timelike [9], in contrast with the naked singularities of inhomogeneous dust collapse, which are typically null). However, all such models belong to a very special subclass of solutions with tangential stresses, the so-called Einstein cluster. This is a system of counter-rotating particles, in which the stress is generated by angular momentum. From the physical point of view, the Einstein cluster is interesting since it mimics the effects of rotation without introducing deviations from spherical symmetry. However, although such a system can be formally obtained by the choice of a particular function of state within the general exact solution, this function is essentially a “centrifugal potential” and therefore does not satisfy the physical characteristics which are typical of the state equations of matter continua, like those to be expected in strongly collapsed matter states (e.g., in neutron stars). As a consequence, the results obtained for the Einstein cluster cannot be straightforwardly extended to the general case of tangential stresses, which is of course interesting in its own right.

Thus motivated, in the present paper we carry out a complete analysis of the final state of gravitational collapse with tangential stresses for (marginally bound) realistic matter configurations. We show that the final state depends on the first non-vanishing term of the Taylor expansion of the initial density distribution near the center (as happens to be the case in dust models) and on the first non-vanishing term of the Taylor expansion of another function which carries all the relevant information coming from the choice of the material. It follows that the final fate of the collapse depends on the choice of two integers, n and k , giving the order of these two expansions, respectively. As we shall see below, there is a critical value for k , above which the tangential pressure effects are negligible, and the endstate of collapse is indistinguishable from that of dust models. Below the critical value, however, tangential stresses come into play, and all the configurations that would otherwise end up in a black hole, terminate in a singularity that is at least locally visible. At the

critical parameter, a transition behavior is observed, wherein the singularity may be visible, depending on the details of the initial data. Both homogeneous and inhomogeneous density evolutions are examined in detail, and they are found to be qualitatively equivalent.

This paper is organized as follows. In Sec. II, we present a short but self-consistent account to recall the physical framework and to prepare the needed mathematical formalism. The results are given in Sec. III, where the structure of the spectrum of end states is presented and discussed. The paper ends with concluding remarks in Sec. IV.

Geometrized units, in which $G=c=1$, are used throughout.

II. SPHERICAL COLLAPSE WITH TANGENTIAL STRESSES

A. The general solution

In this section we review briefly the main properties of spherically symmetric gravitating systems with vanishing radial stresses (for details, see [6,7]).

The mathematical structure of the field equations is simplified by the fact that the Misner-Sharp mass is conserved. This actually allows complete integration in the so-called mass-area coordinates, in which the line element reads

$$ds^2 = -K^2 \left(1 - \frac{2m}{R} \right) dm^2 + 2 \frac{KE}{uh} dR dm - \frac{1}{u^2} dR^2 + R^2 d\Omega^2. \quad (2.1)$$

Here K , u and h are functions of R and m , and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the canonical metric of the unit two-sphere. The function u is the modulus of the velocity of the collapsing shells and satisfies

$$u^2 = -1 + \frac{2m}{R} + \frac{E^2}{h^2}, \quad (2.2)$$

while

$$K = g(m) + \int G(m, R) dR, \quad (2.3)$$

where

$$G(m, R) := \frac{h}{RE} \left[1 + \frac{R}{2} \left(\frac{E^2}{h^2} \right)_{,m} \right] \left(-1 + \frac{2m}{R} + \frac{E^2}{h^2} \right)^{-3/2}. \quad (2.4)$$

The functions $g(m)$ and $E(m)$ are arbitrary. The quantity $h(m, R)$ is a measure of the internal energy stored in the material, and can thus be regarded as an effective equation of state (the physical characterization of this function will be given below). It can be shown that the energy density ϵ and the tangential stress Π , can be written in terms of h as

$$\epsilon = \frac{h}{4\pi u K E R^2}, \quad (2.5)$$

$$\Pi = -\frac{R}{2h} \frac{\partial h}{\partial R} \epsilon. \quad (2.6)$$

These formulas show that the tangential stress vanishes whenever the function h is independent of R . In such a case, the material is not sustained by any internal stress and the line element (2.1) reduces to the one describing spherically symmetric dust in mass-area coordinates [13]. Since, in all formulas, only the ratio of E and h appears, the value of h along an arbitrary curve $R=R_0(m)$ can be rescaled to unity, so that, in particular, the dust solutions can be characterized by $h=1$. The function $R_0(m)$ plays the same role as that played by the initial mass distribution in the standard comoving coordinates [the inverse transformation from mass label to comoving label being $r=R_0(m)$], while the function $E(m)$ is the specific binding energy function.

The above recalled structure shows that a solution with tangential stresses is identified—modulo gauge transformations—by a triplet of functions $\{g, E, h\}$. This parametrization contains the dust spacetimes as the subset $\{g, E, 1\}$. In this way, we can construct in a mathematically precise fashion, a comparison between dust and tangential stress evolutions, by comparing the end state of the dust collapse ($\{g, E, 1\}$ with chosen g and E) with the end states of the tangential stress solutions $\{g, E, h\}$ with the same g and E , and different choices of the equation of state h .

The physical singularities of the spacetime described by the metric (2.1) correspond to infinite energy density, and are given by $R=0$ or $K=0$. At $R=0$, the shells of matter collapse to zero proper area, thereby leading to shell-focusing singularities, whereas the vanishing of K implies intersection of different shells of matter, corresponding to shell-crossing singularities. If occurring, such singularities are gravitationally weak and we shall not deal with them in the present paper.

Non-central shells which become singular ($R=0$ and $m \neq 0$) are always spacelike—thus covered—in spherical spacetimes with radial stresses, as a consequence of mass conservation [7]; only the central singularity $R=m=0$ can be visible, since at $R=m=0$, the apparent horizon and the singularity form simultaneously. The visibility is determined by the existence (or lack thereof) of outgoing null geodesics with past end point at the singularity. To investigate the existence of such geodesics, we use the general method originally developed by Joshi and Dwivedi [14]. One defines the quantity $x = R/2m^\alpha$ (where $1/3 < \alpha \leq 1$) and observes that, at the singularity, both R and m vanish. Using l'Hôpital's rule, one can arrive at an equation which, in the present model, takes the form

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{m^{(3/2)(1-\alpha)}}{2\alpha} \left[-\frac{R_{0,m} h(m, R_0)}{E(m) u(m, R_0)} + \int_{R_0}^{2m^{\alpha x}} G dR \right] \sqrt{\left(-1 + \frac{E^2}{h^2} \right) m^{\alpha-1} + \frac{1}{x}} \\ \times \left(\frac{E}{h} - \sqrt{-1 + \frac{E^2}{h^2} + \frac{m^{1-\alpha}}{x}} \right). \quad (2.7)$$

If this equation admits real positive-definite roots, i.e., at least one finite, positive solution x_0 exists for some α , then this root represents the tangent to an outgoing null geodesic meeting the singularity in the past, which is therefore at least locally naked (covered otherwise).

B. Equation of state, regularity, and physical reasonability conditions

As recalled in the previous section, the choice of the matter model in a solution with tangential stresses corresponds to the choice of the function h . This is equivalent to the choice of an equation of state like, e.g., the barotropic equation of state connecting density and pressure in the case of a perfect fluid. The choice of h is thus restricted by physical considerations, as follows [15].

The internal energy per unit volume of the material (ϵ , say) is given by $\epsilon = \rho_b h$, where ρ_b is the baryon number density. If the material is physically viable, the internal energy has to coincide with the number density in a (locally) relaxed state of the continuum. This obviously means, as in the classical (i.e., non-relativistic) mechanics of continua, that h must be a positive, convex function having a minimum in the locally relaxed state. Without loss of generality, we take the value of the minimum equal to one. Since all deformations are described “gravitationally” in the frame we are working with, the state of local relaxation corresponds to the flat space values of the metric components. It follows that the most general physically valid equation of state has the form

$$h(m, R) = 1 + \beta(m)(R_0 - R)^2 + h_2(m, R), \quad (2.8)$$

where $h_2(m, R)$ goes to zero more rapidly than $(R_0 - R)^2$, as R tends to R_0 . The function $\beta(m)$ is positive and models the “strength” of the tangential pressure. With the scaling $R_0(m) = r$, the squared factor on the right-hand side lies in the interval $[0, r^2]$, where the lower and upper limits are realized on the initial and singular slices, respectively. Since we are interested only in the behavior near the central singularity, it follows that, as far as the causal structure of the singularity is concerned, we can consider the function h to be of the form

$$h(m, R) = 1 + \beta_k m^{k/3} (R_0 - R)^2, \quad (2.9)$$

where k must be positive in order to assure regularity at the center, and the positive constant β_k measures the effects of tangential stresses. The case of dust collapse corresponds to $\beta_k = 0$, while for β_k nonzero the first nonvanishing term of the Taylor expansion of the tangential stress near the center is of order k .

Having chosen a suitable matter model to address the issue of cosmic censorship, one must also ensure that several other reasonability conditions are satisfied.

First of all, the spacetime must have a regular center (so that the singularity forming at $r=0$ is a true dynamical singularity). This requires the functions K , u , and E/h to be smooth and bounded away from zero. Further, we require the collapse to proceed from a regular initial slice, where the metric and the exterior curvature are continuous, the physical

quantities are bounded, and there are no trapped surfaces. As we have seen, initial data consist of two independent quantities g and E , which can be equivalently described by the initial mass $[m(r)$, i.e., $R_0(m)]$ and the initial velocity distribution $f := E^2 - 1$. In what follows, we take $f=0$, i.e., we consider marginally bound configurations. Regularity on the initial surface implies that

$$m(r) = F_0 r^3 + F_n r^{n+3} + \mathcal{O}(r^{n+4}), \quad (2.10)$$

where F_n denotes ($4\pi/3$ times) the first non-vanishing derivative of the initial energy density profile at the origin. Positivity of mass requires $F_0 > 0$, and $F_n < 0$ for any realistic configuration, where the energy density decreases away from the center.

We want to exclude the presence of trapped surfaces on the initial slice. This requires $2m < R$ on the $R_0(m) = r$ surface:

$$\left(\frac{2m}{R}\right)_{R=r} = \frac{2m(r)}{r} = 2F_0 r^2 + \mathcal{O}(r^{n+2}) < 1, \quad (2.11)$$

where the second equality holds only near the origin. The first equality imposes a maximum size on the matter distribution, for a given density profile. Near the origin, the requirement $2m/R < 1$ is always satisfied on the initial slice, for $0 \leq r < 1/\sqrt{2F_0}$.

Finally, and in accord with the spirit of the cosmic censorship conjecture [16], we require that the matter content obeys the weak energy condition (WEC):

$$\epsilon > 0, \quad \epsilon + \Pi > 0. \quad (2.12)$$

From Eq. (2.5), one sees that the first inequality is satisfied provided $h > 0$ and $K < 0$ (recall that $u < 0$, for the collapsing situation we are interested in). From Eq. (2.9), it follows that $\partial_R(\ln h) < 0$, and hence the tangential stress is positive, which is a sufficient condition for the second inequality (positivity of Π implies that also the strong energy condition, $\epsilon + 2\Pi \geq 0$, is automatically satisfied here).

C. The final state of collapse

Dust models do not admit globally regular solutions, i.e., singularity-free solutions. This is not necessarily true for general matter fields, even within spherical symmetry. Examples of globally regular solutions in spherical symmetry include certain classes of perfect fluid models, which evolve to an eternally oscillating configuration from regular initial data (see, e.g., [17]), and solitonic solutions of the massive Einstein-Klein-Gordon system [18]. Accordingly, it is important to understand the qualitative behavior of the dynamics, from a given set of regular initial data, before addressing the causal nature of the final state.

The equation of motion (2.2) for the collapsing shells can be rewritten as

$$u^2 = \frac{Z(R, R_0)}{R[1 + \beta(m)(R_0 - R)^2]}, \quad (2.13)$$

where the “effective potential” Z is defined by

$$Z(R, R_0) := 2m + \beta(m)(2m - R)(R_0 - R)^2. \quad (2.14)$$

This function may be regarded as the analogue of the Newtonian effective potential governing the motion of the *fixed* shell R_0 , wherein the allowed regions of the motion correspond to $Z \geq 0$. Setting $R = \zeta(t, R_0)R_0$, to the lowest order in R_0 we obtain

$$2F_0[1 + \beta_k m^{k/3}(R_0 - R)^2] \geq \beta_k m^{k/3} \zeta(t). \quad (2.15)$$

This condition always holds, independently of the choice of k , as m goes to zero. Therefore, there is always a region of initial data leading to continued gravitational collapse.

We now would like to check whether the central point ($m=0$) eventually gets trapped. Since m is conserved for any given shell, this can be done by analyzing the dynamics of shells near the center, along curves $R = \lambda m$ with $\lambda > 2$ [12]. The requirement $Z \geq 0$ now reads

$$2 \geq \beta_k m^{k/3}(R_0 - \lambda m)^2(\lambda - 2). \quad (2.16)$$

Clearly, near the center this inequality always holds, which implies that non-central shells inevitably become singular. This is in contrast with the Einstein cluster case, where rotation has the effect that all the shells near the center one remain regular for all times [10]. In the present case, instead, the apparent horizon inevitably forms, and therefore the central singularity may or may not be visible, depending on the existence of geodesics meeting this singularity in the past.

III. THE SPECTRUM OF END STATES

A. The root equation

The analysis of the root equation for gravitational collapse with tangential stresses is non-trivial, because of the integral appearing on the right-hand side of Eq. (3.5), which introduces a “non-local” behavior. Indeed, as we have seen, only the case of the Einstein cluster has been analyzed so far.

To check whether the root equation has positive-definite solutions, we observe that, although the integral appearing in square brackets cannot be carried out in terms of elementary functions, its behavior near the center can be analyzed as follows. First, let us denote

$$I_1(m; \alpha) := \int_{R_0}^{2m^{\alpha x}} G(m, R) dR, \quad (3.1)$$

where $G(m, R)$ is defined by Eq. (2.4). Due to the mean value theorem, there exists $\chi(m) \in (R_0, 2m^{\alpha x})$ such that

$$I_1(m; \alpha) := (2m^{\alpha x} - R_0)G(m, \chi(m)). \quad (3.2)$$

Since $R_0 \sim m^{1/3}$ as m goes to zero, $1/3 < \alpha \leq 1$, and x is positive-definite, it follows that both $m^{-1/3}\chi(m)$ and $m^{\alpha}/\chi(m)$ are also positive-definite as $m \rightarrow 0$. One can thus evaluate the right hand side of Eq. (3.2)—using Eq. (2.4)—as follows:

$$I_1(m; \alpha) = m^{-1} \frac{(2m^{(3\alpha-1)/3}x - F_0^{-1/3} + \dots)}{2\sqrt{2}} \left(\frac{\chi}{m^{1/3}} \right)^{1/2} \left[1 - \frac{\chi}{m^{1/3}} \left(\frac{\beta[F_0^{-1/3} - (\chi/m^{1/3})]^2}{2[1 + \beta(R_0 - \chi)^2]} \right) \right]^{-3/2} \\ \times \left[1 - \frac{\chi}{m^{1/3}} \left(\frac{(m\beta_{,m})[F_0^{-1/3} - (\chi/m^{1/3})]}{2[1 + \beta(R_0 - \chi)^2]^2} \right) - \frac{\chi}{m^{1/3}} \frac{(\beta F_0^{-1/3})[F_0^{-1/3} - (\chi/m^{1/3})]}{[1 + \beta(R_0 - \chi)^2]^2} \right] [1 + \beta(R_0 - \chi)]^{1/2}, \quad (3.3)$$

where the dots stand for terms of higher order in positive powers of $m^{n/3}$. Since $\beta(0) = 0$, the divergence is driven by $\mathcal{O}(m^{-1})$, with all the other terms remaining finite. Therefore, the limit

$$I_2(m; \alpha) := \lim_{m \rightarrow 0} \int_{R_0}^{2m^{\alpha x}} m G(m, R) dR, \quad (3.4)$$

is convergent. Using Lebesgue’s dominated convergence theorem, we can expand the integrand near the center ($m=0$) in leading powers of m , and integrate successive terms [19]. It follows that the root equation, for a general density distribution [given by Eq. (2.10)], with the equation of state (2.9), can be written as

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{m^{(3/2)(1-\alpha)}}{\alpha} \\ \times \left(\Theta_n m^{-1+n/3} + B_k m^{-1+k/3} + \frac{2m^{3(\alpha-1)/2}}{3} x^{3/2} + \dots \right) \\ \times \left(\frac{1}{\sqrt{x}} - \frac{m^{(1-\alpha)/2}}{x} \right), \quad (3.5)$$

where

$$\Theta_n := - \frac{nF_n}{18\sqrt{2}F_0^{(2n+9)/6}}, \quad (3.6)$$

TABLE I. The end state of collapse for inhomogeneous collapse with tangential pressures. See text for details.

Initial data	$n=1$	$n=2$	$n=3$	$n \geq 4$
$k=1$	$\frac{-21F_1 + 8\beta_1 F_0^{1/3}}{84\sqrt{2}F_0^{11/6}}$	$\frac{\sqrt{2}\beta_1}{21F_0^{3/2}}$	$\frac{\sqrt{2}\beta_1}{21F_0^{3/2}}$	$\frac{\sqrt{2}\beta_1}{21F_0^{3/2}}$
$k=2$	$\frac{-F_1}{4\sqrt{2}F_0^{11/6}}$	$\frac{-105F_2 + 22\beta_2 F_0^{2/3}}{420\sqrt{2}F_0^{13/6}}$	$\frac{11\beta_2}{210\sqrt{2}F_0^{3/2}}$	$\frac{11\beta_2}{210\sqrt{2}F_0^{3/2}}$
$k=3$	$\frac{-F_1}{4\sqrt{2}F_0^{11/6}}$	$\frac{-F_2}{4\sqrt{2}F_0^{13/6}}$	$\gamma_{33} > \gamma_c$	$\gamma_{43} > \gamma_c$
$k \geq 4$	$\frac{-F_1}{4\sqrt{2}F_0^{11/6}}$	$\frac{-F_2}{4\sqrt{2}F_0^{13/6}}$	$\gamma_{34} > \gamma_c$	no positive real roots

$$B_k := \frac{\sqrt{2}(9+k)\beta_k}{945F_0^{3/2}}. \quad (3.7)$$

Standard results from polynomial theory can be used to show that this quartic has real and positive roots if $\gamma > \gamma_c$, where [3]

$$\gamma_c := (26 + 15\sqrt{3})/4. \quad (3.12)$$

B. The dust limit

In order to better visualize the effects of tangential stresses in spherical dust collapse, it is important to recall briefly how to recover the (very well known) dust case from our general set-up of the previous section (for a complete review and list of references see [3]). This is easily done by setting $\beta(m)=0$, and thus $\beta_k=0$. The root equation becomes then

$$x_0 = \lim_{m \rightarrow 0} \frac{1}{\alpha} \left[\Theta_n m^{(3+2n-9\alpha)/6} + \frac{x_0^{3/2}}{3} + \dots \right] x^{-1/2} \times [1 - x^{-1/2} m^{(1-\alpha)/2}]. \quad (3.8)$$

It is straightforward to check that for $\alpha \neq \alpha_n \equiv \frac{1}{3}(1+2n/3)$ the root equation does not admit any positive-definite solution: if $\alpha < \alpha_n$, then $x_0=0$, and for $\alpha > \alpha_n$ the right-hand side of Eq. (3.8) diverges in the $m \rightarrow 0$ limit. One must therefore have $\alpha = \alpha_n$. In this case, the equation reduces to

$$x_0^{3/2} = \frac{1}{\alpha} \left(\Theta_n + \frac{x_0^{3/2}}{3} \right) [1 - x_0^{-1/2} \lim_{m \rightarrow 0} m^{(3-n)/9}]. \quad (3.9)$$

Clearly, we must have $n \leq 3$, for $x_0 \in (0, +\infty)$. For $n < 3$, we obtain

$$x_0 = \left[\left(1 + \frac{3}{2n} \right) \Theta_n \right]^{2/3}, \quad (3.10)$$

and the singularity is locally visible. For $n=3$, after a little algebra, one obtains the quartic equation (where $Z \equiv \sqrt{x_0}$):

$$2Z^4 + Z^3 + \gamma(1-Z) = 0, \quad (3.11)$$

with

$$\gamma = \Theta_3 = F_3 F_0^{-15/6} / (2\sqrt{2}).$$

C. Structure of the end states

For $\beta_k \neq 0$, the root equation is given by Eq. (3.5). It is immediately seen that one can always choose $1 \leq n \leq 3$ and/or $1 \leq k \leq 3$, in such a way that, for suitable $1/3 < \alpha \leq 1$, this equation gives a finite, algebraic condition. It then follows that in all such cases the singularity can be naked.

For $n < 3$ or $k < 3$ the singularity is naked, since a positive root always exists (we omit here tedious, but straightforward calculations; the values of the roots are reported in Table I).

For $k \geq 4$, the effects of tangential stresses are negligible, and the endstate is exactly the same as that in dust collapse, with the shape of the initial central density profile determining the existence of positive-definite roots, and thus local visibility. The very same can be said, however, about the effects of inhomogeneity for $n \geq 4$, since *all* such configurations would lead to black holes, while now they all terminate in a visible singularity if $k < 3$.

A transitional (“critical”) behavior is observed at the three entries forming the “boundary” of the $n < 3, k < 3$ region. Here, naked singularities occur if a quartic equation has positive solution(s). This quartic is identical in form to that holding for dust, namely

$$2Z^4 + Z^3 + \gamma_{nk}(1-Z) = 0, \quad (3.13)$$

where γ_{nk} takes the values

$$\gamma_{33} = \gamma + \delta, \quad (3.14)$$

$$\gamma_{34} = \gamma, \quad (3.15)$$

$$\gamma_{43} = \delta, \quad (3.16)$$

with

$$\delta := \frac{4\sqrt{2}\beta_3}{105F_0^{3/2}}. \quad (3.17)$$

Therefore, at each “boundary,” naked singularities are formed if the corresponding quantity γ_{nk} is greater than the quantity γ_c defined in Eq. (3.12).

D. The homogeneous case

The homogeneous dust solution is the well-known Oppenheimer-Snyder spacetime [20] (the seminal paradigm for black hole formation, for over sixty years), whose dynamical evolution results in a Schwarzschild black hole, with an interior, spacelike (hence covered) singularity. To evaluate the root equation for this case, we simply set $F_n=0$ in Eq. (3.8), obtaining

$$x_0^{1/2} = \lim_{m \rightarrow 0} \frac{1}{3\alpha} [x^{1/2} - m^{(1-\alpha)/2}]. \quad (3.18)$$

Clearly, we need $\alpha \leq 1$, or else the right-hand side of the above equation diverges negatively in the $m \rightarrow 0$ limit. For $\alpha=1$, we obtain $x_0^{1/2}(2x_0^{1/2}+1)=0$, which has no positive-definite solution, and the singularity is therefore covered. For $\alpha < 1$, the x_0 terms drop out, and a self-consistent solution exists if $\alpha=1/3$, which is *not* allowed. Hence, the singularity is always covered, as expected.

This example provides a particularly simple test-bed for studying the effects of tangential pressure on the final state of collapse. In the presence of tangential stresses, from Eq. (3.5) with $\Theta_n=0$, we obtain

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{1}{\alpha} \left[B_k m^{(3+2k-9\alpha)/6} + \frac{x^{3/2}}{3} + \dots \right] \left[\frac{1}{\sqrt{x}} - \frac{m^{(1-\alpha)/2}}{x} \right]. \quad (3.19)$$

A curious phenomenon occurs here. In fact, the above equation is formally identical to the root equation for *inhomogeneous* dust [cf. Eq. (3.8) above] with k and B_k playing the role of n and θ_n respectively. Clearly, in the leading term we can always choose $1 \leq k \leq 3$, such that $1/3 < \alpha \leq 1$, and the singularity may thus be visible. Again, there is a clear transition at $k=3$, below which (stronger tangential stresses) the singularity is always visible, and above which (weaker tangential stresses) is always covered. At the transitional value, $k=3$, we have again to solve a quartic of the kind (3.13) with parameter equal to δ , so that naked singularities occur if $\delta > \gamma_c$. The final state of collapse is summarized in Table II, below.

IV. CONCLUDING REMARKS

We have examined here the effects of tangential stresses on the final state of (marginally bound) collapse. Since the nature of the singularity turns out to depend only on the Taylor expansion of the data and of the state function near

TABLE II. The end state of homogeneous collapse with tangential pressure.

Initial data	Root $(x_0)^{3/2}$	Singularity
$k=1$	$\frac{\sqrt{2}\beta_1}{21F_0^{3/2}}$	visible
$k=2$	$\frac{11\beta_2}{210\sqrt{2}F_0^{3/2}}$	visible
$k=3$	$\delta > \gamma_c$	transition
$k \geq 4$	no real positive roots	covered

the center, it is possible to classify the final state of all the solutions that have a physically reasonable equation of state. This classification can be done in terms of the “strength” of the tangential stress near the center. If this strength is above a certain threshold, the tangential pressure dominated collapse always leads to a locally naked singularity, independently of the choice of the initial data. If, instead, the tangential pressure is sufficiently weak, then the final state is the same as that occurring in dust collapse, where the shape of the initial density profile near the origin fully determines the visibility of the singularity.

These results actually disprove a conjecture, recently put forward by one of us [7], the proposal of which, roughly speaking, was that the space of the data leading to naked singularities in spherical collapse could not be “larger” than that of dust initial data leading to the same singularities. Contrary to this expectation, the present results are consistent with quite a different scenario: they support the notion that the strong version of the cosmic censorship conjecture is likely to be violated for realistic matter distributions in a way that can safely be described as *common*, insofar as the matter content is concerned. Indeed, our analysis shows that naked singularities are *not* an artifact of dust models, or a byproduct of “toy” models constructed without giving the equation of state of the collapsing matter, or, finally, a spurious outcome of unnatural effects induced by rotation in spherical symmetry.

There remains the possibility that such naked singularities are an artifact of spherical symmetry by itself, although both numerical and analytical results suggest that dynamical curvature singularities may persist in non-spherical collapse [21–23].

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